2020/TDC/ODD/SEM/MTMH–102/255

TDC Odd Semester Exam., 2020
held in July 2021

MATHEMATICS

(Honours)

(1st Semester)

Course No. : MTMH–102

(Differential Calculus and Integral Calculus—I)

Full Marks : 50
Pass Marks : 17

Time : 2 hours

The figures in the margin indicate full marks for the questions

Answer **five** questions, selecting **one** from each Unit

UNIT—I

1. *(a)* Using - definition, show that

$$\lim_{x \quad 0} x^2 \sin\frac{1}{x} \quad 0$$

3

(b) State Cauchy's criterion for existence of a limit. Using this, show that $\lim_{x \quad 0} \cos\frac{1}{x}$ does not exist. 2+2=4

(c) Show that the function f defined by

$$f(x) \quad \begin{cases} \dfrac{e^{\frac{1}{x}} \quad e^{\frac{1}{x}}}{e^{\frac{1}{x}} \quad e^{\frac{1}{x}}}, & \text{when } x \quad 0 \\ 0 \quad , & \text{when } x \quad 0 \end{cases}$$

is not continuous at $x \quad 0$. 3

2. *(a)* If a function is differentiable at a point, show that it is continuous thereat. Show with an example that the converse is not true. 3+1=4

(b) If

$$f(x) \quad \begin{cases} x^2 \cos\frac{1}{x} \quad , & \text{when } x \quad 0 \\ 0 \quad , & \text{when } x \quad 0 \end{cases}$$

find $f \ (0)$. 2

(c) Show that the function f defined by

$$f(x) \quad \begin{cases} x^2 \sin\frac{1}{x} \quad , & \text{when } x \quad 0 \\ 0 \quad , & \text{when } x \quad 0 \end{cases}$$

is differentiable at $x \quad 0$ and its first derivative is not continuous thereat. 4

UNIT—II

3. *(a)* If $y = x^{2n}$, where n is a positive integer, show that

$$y_n = 2^n\{1\cdot 3\cdot 5\cdots(2n-1)\}\, x^n$$ 2

(b) If $u = \sin ax + \cos ax$, show that

$$u_n = a^n\sqrt{\{1+(-1)^n \sin 2ax\}}$$ 3

(c) State and prove Lagrange's mean value theorem. 5

4. *(a)* If $y = e^{a\sin^{-1}x}$, prove that

$$(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2+a^2)y_n = 0$$ 4

(b) Evaluate

$$\lim_{x\to 0}(\cos x)^{\frac{1}{x^2}}$$ 3

(c) Show that

$$\sin x > x - \frac{x^3}{6}$$

if $0 < x < \frac{\pi}{2}$. 3

UNIT—III

5. *(a)* Expand $\sin x$ in infinite series of powers of x stating the validity conditions. 3

(b) Show that at any point on the curve

$$x^{m+n} = k^{m-n}.y^{2n},$$

the mth power of subtangent varies as the nth power of subnormal. 4

(c) Show that the portion of the tangent at any point on the curve

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$$

intercepted between the axes is of constant length. 3

6. *(a)* Prove that the subnormal at any point of a parabola is of constant length and the subtangent varies as the abscissa of the point of contact. 2+2=4

(b) Find the condition that the conics $ax^2 + by^2 = 1$ and $a_1x^2 + b_1y^2 = 1$ shall cut orthogonally. 3

(c) Show that the normal at the point $\theta = \frac{\pi}{4}$ on the curve

$$x = 3\cos\theta - \cos^3\theta,\ y = 3\sin\theta - \sin^3\theta$$

passes through the origin. 3

UNIT—IV

7. (a) If $u = \log(x^2 + y^2)$, prove that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$ 3

(b) If

$$u = \tan^{-1}\frac{x^3 + y^3}{x - y}$$

prove that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 2u$$ 3

(c) Show that the semivertical angle of a cone of maximum volume and given slant height is $\tan^{-1}\sqrt{2}$. 4

8. (a) If $u = f(x + ct) + g(x - ct)$, show that

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}$$ 2

(b) Find the radius of curvature for the curve $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$ at $\theta = 0$. 4

(c) If

$$u = \log(x^3 + y^3 + z^3 - 3xyz)$$

show that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = -\frac{3}{(x + y + z)^2}$$ 4

UNIT—V

9. (a) Show that

$$\int_0^1 \frac{\log(1 + x)}{1 + x^2}dx = \frac{\pi}{8}\log 2$$ 4

(b) Show that

$$\int_0^{\pi/2} \frac{x\,dx}{\sin x + \cos x} = \frac{\pi}{2\sqrt{2}}\log(1 + \sqrt{2})$$ 4

(c) Evaluate

$$\int_0^{\pi/2} \sin^9 x \cos^{12} x\,dx$$ 2

10. (a) Show that

$$\int_0^{\pi/2} \frac{dx}{3 + 5\cos x} = \frac{1}{4}\log 3$$ 4

(b) If

$$u_n = \int_0^{\frac{\pi}{2}} x^n \sin x\,dx$$

show that

$$u_n + n(n-1)u_{n-2} = n\left(\frac{\pi}{2}\right)^{n-1}$$ 4

(c) Show that

$$\int_0^{\frac{\pi}{2}} \log \tan x\,dx = 0$$ 2

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